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SOLAR HEATING OF A ROTATING SPHERICAL SPACE VEHICLE

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Abstract—This paper is concerned with the temperature distribution of a thin-walled spherical satellite subjected to solar radiation. The satellite rotates about an axis perpendicular to the incident radiation. The linearized differential equation of the temperature distribution is solved in terms of Legendre polynomials and associated Legendre functions. The results demonstrate the role of the angular velocity in the temperature distribution. A numerical example is included and an error analysis is performed in order to evaluate errors due to the linearization process.

NOMENCLATURE

a,	average absorptivity of the wall material;
a',	$-\dfrac{aK_s r_0^2}{T_0 ks}$, a dimensionless group;
b',	$\dfrac{\rho c_p r_0^2 \omega}{k}$, a dimensionless group;
B_n,	coefficients of expansion of the radiation input function;
B_n^m,	coefficients of expansion of the radiation input function;
c',	$\dfrac{4\sigma e r_0^2 T_0^3}{ks}$, a dimensionless group;
c_p,	specific heat of the wall material;
C_n^m,	coefficients of expansion of the radiation input function;
D_n,	coefficients of solution of the differential equation;
D_n^m,	coefficients of solution of the differential equation;
e,	average emissivity of the wall material;
E_n^m,	coefficients of solution of the differential equation;
f,	radiation input function in the r, ψ, ϕ co-ordinate system;
F,	radiation input function in the r, θ, ϕ co-ordinate system;
k,	thermal conductivity of the wall material;
K_s,	energy per unit area and time received by a plane normal to the parallel radiation;
$P_n(\cos\phi)$,	Legendre polynomial of degree n;
$P_n^m(\cos\phi)$,	associated Legendre function of order m and degree n;
r, θ, ϕ,	spherical co-ordinates (radius, longitude, co-latitude) fixed with respect to the incoming parallel radiation;
r, ψ, ϕ,	spherical co-ordinates (radius, longitude, co-latitude) fixed with respect to the rotating satellite;
r_0,	radius of the satellite;
R,	$\dfrac{r_0^2}{ks}(4\sigma e a^3 K_s^3)^{1/4}$, a dimensionless parameter of the solution;
s,	wall thickness;
S,	spherical surface area in the r, ψ, ϕ co-ordinate system;
t,	time variable in the r, ψ, ϕ co-ordinate system;
T,	local satellite temperature (absolute) in the r, θ, ϕ co-ordinate system;
$\hat{T}$,	non-dimensional variation from the average temperature;
$\bar{T}$,	$\hat{T} + \frac{1}{4}$;
T_0,	reference temperature (absolute);

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$\tilde{T}$, true satellite temperature (absolute) that would be found from the solution of the non-linear problem;
x, $\cos\phi$;
X, Y, Z, co-ordinate axes fixed with respect to the incoming parallel radiation;
α, $\frac{k}{\rho c_p}$, thermal diffusivity;
β, coefficient of the solution of the differential equation;
γ, coefficient of the solution of the differential equation;
$\delta_{\max}$, angle of shift of positions of maximum temperature;
$\delta_{\min}$, angle of shift of positions of minimum temperature;
ϵ_Q, error in overall heat balance induced by the linearized solution;
ϵ_T, local error in the temperature induced by the linearized solution;
$\bar{\epsilon}_T$, weighted average of ϵ_T over the entire satellite;
ρ, density of the wall material;
σ, Stefan–Boltzman constant;
τ, local satellite temperature (absolute) in the r, ψ, ϕ co-ordinate system;
ω, angular velocity of the satellite;
ω^*, $\frac{\alpha}{r_0^2}$, "thermal angular velocity";
ω_0, $\frac{\omega}{\omega^*}$, a dimensionless parameter of the solution.

INTRODUCTION

THE design of space vehicles and satellites has stimulated an interest to find the surface temperature distribution for various geometrical shapes exposed to solar radiation. A brief review of literature can be found in reference [1]. The references included in this paper pertain directly to the case under consideration.† In this study, a thin-walled spherical satellite, rotating about an axis of symmetry, is considered. Furthermore, this axis is perpendicular to a line connecting the center of the radiation source with the center of the satellite.

A quasi steady state exists when a balance has been achieved between the radiant heat absorbed by the satellite and the heat re-radiated into space. The characteristic of this state is that each point on the rotating shell will experience the same temperature upon returning to any given position which is fixed with respect to the radiation source. It is the temperature distribution associated with this quasi steady state which will be investigated.

During the space flight of the satellite, the external convection heat transfer will be neglected. Furthermore, it is assumed that the satellite design is such that all heat transferred internally can be neglected. The shell thickness will be considered as sufficiently thin for neglection of temperature gradients in the radial direction; thus heat is conducted only in directions tangent to the shell surface. It is assumed that the distance separating the source and satellite surface is large, so that the entire satellite surface radiates to space. Also, the effective temperature of space is assumed to be quite small as compared with the satellite temperature, so that radiation from the satellite is proportional to the fourth power of satellite temperature alone.

1. DIFFERENTIAL EQUATION OF THE TEMPERATURE DISTRIBUTION

1.1. *The energy balance for a volume element*

A spherical shell of radius r_0 and thickness s is fixed in a spherical co-ordinate system of r (radius), ψ (longitude), ϕ (co-latitude). The sphere and its co-ordinate system rotate, with angular velocity ω, about the Z-axis of an X, Y, Z co-ordinate system. The incoming radiation is normal to the XZ-plane (Fig. 1).

The energy balance for a volume element of the shell is

$$q_i\,\mathrm{d}S\,\mathrm{d}t - q_0\,\mathrm{d}S\,\mathrm{d}t - \mathrm{d}Q_c\,\mathrm{d}t = \rho c_p \frac{\partial \tau}{\partial t}\,\mathrm{d}V\,\mathrm{d}t \tag{1.1}$$

where

$q_i(\psi, \phi, t) =$ radiant heat absorbed at the external face of the volume element per unit area and time.

† A very recent contribution to the field is given in reference [5].

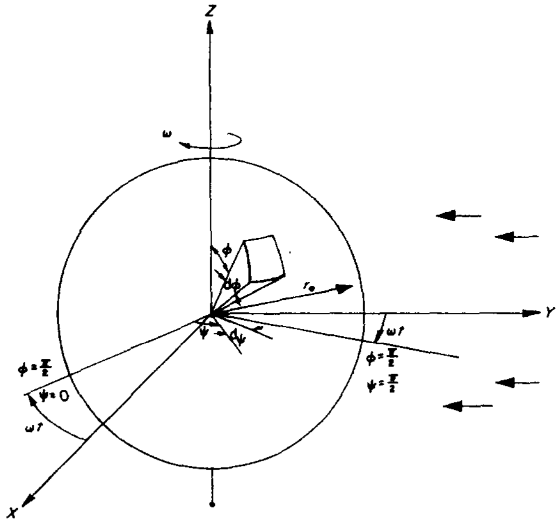

FIG. 1. Satellite co-ordinate system.

$q_0(\psi, \phi, t)$ = heat radiated into space from the external face of the volume element per unit area and time.

$dQ_c(\psi, \phi, t)$ = net conducted heat outflow from the volume element per unit time.

$\tau(\psi, \phi, t)$ = surface temperature (absolute) of the shell.

$\rho c_p \dfrac{\partial \tau}{\partial t}$ = heat stored per unit volume and time.

dS = surface area differential.

dV = volume differential.

dt = time differential.

(a) Absorbed radiant energy, q_i, is

$$q_i = aK_s f(\psi, \phi, t) \tag{1.2}$$

where

a = average (with respect to wave length and angle of incidence) absorptivity of the satellite material.

K_s = energy per unit area and time received from the radiating source by a plane normal to the radiation.

$f(\psi, \phi, t)$ = a function dependent on the geometry of the system (to be defined later).

(b) The energy radiated from the satellite into space, q_0, is found from the Stefan–Boltzman law,

$$q_0 = \sigma e \tau^4 \tag{1.3}$$

where

σ = Stefan–Boltzman constant,

e = average total hemispherical emissivity of the satellite material for the spectrum of wave length radiated.

(c) The net conducted heat outflow from the volume element, dQ_c, consists of two components. As indicated in Fig. 2, Q_1 is the heat conducted in a direction normal to the area dS_1, and Q_2 is the heat conducted in a direction normal to dS_2. Thus,

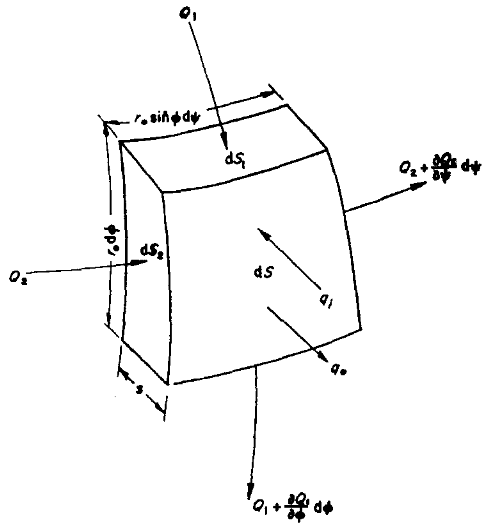

FIG. 2. Heat flow in a spherical volume element.

$$dQ_c = \frac{\partial Q_1}{\partial \phi} d\phi + \frac{\partial Q_2}{\partial \psi} d\psi.$$

From the Fourier law of heat conduction, the geometry of the volume element and the heat balance follows

$$\frac{aK_s r_0^2}{ks} f(\psi, \phi, t) - \frac{\sigma e r_0^2}{ks} \tau^4 + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi}\left(\sin \phi \frac{\partial \tau}{\partial \phi}\right) + \frac{1}{\sin^2 \phi} \frac{\partial^2 \tau}{\partial \psi^2} = \frac{\rho c_p r_0^2}{k} \frac{\partial \tau}{\partial t}. \tag{1.4}$$

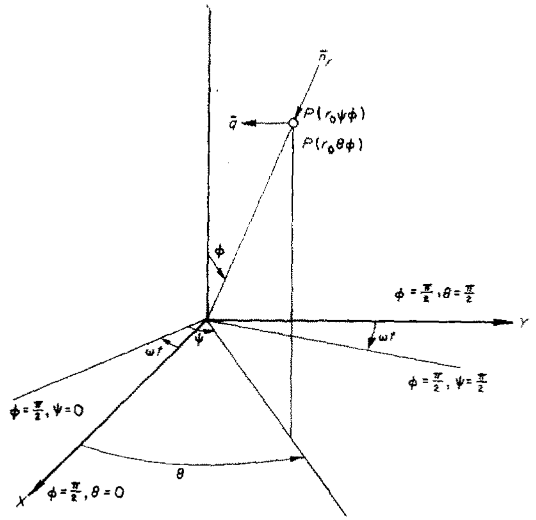

FIG. 3. Moving and fixed co-ordinate systems.

Equation (1.4) is the general differential equation for the local satellite temperature as a function of position and time.

1.2. Co-ordinate transformation

Consider another spherical co-ordinate system r, θ, ϕ which is fixed with respect to the radiation source, and hence is fixed with respect to the X, Y, Z co-ordinate system (Fig. 3). When a balance has been achieved between the absorbed radiation and the heat re-radiated from the satellite, then any point, $P(r_0, \psi, \phi)$ on the satellite will have a temperature which is varying in a periodic manner with time; and each time this point coincides with a fixed position, $P(r_0, \theta, \phi)$, the same temperature will result. Hence, a temperature which is time independent can be associated with every fixed position, $P(r_0, \theta, \phi)$. This is the quasi steady state temperature distribution. Schneider [2] gives the mathematical transformation between the fixed and moving system in Cartesian co-ordinates. Extending to the spherical co-ordinate system used here,

$$\theta = \psi - \omega t \qquad t' = t \qquad \phi = \phi \tag{1.5}$$

where t' is the time variable in the r, θ, ϕ system.

Further, define

$$\tau(\psi, \theta, t) = T(\theta, \phi) \tag{1.6}$$

$$f(\psi, \phi, t) = F(\theta, \phi). \tag{1.7}$$

The general transformations for the first partial derivatives are,

$$\frac{\partial \tau}{\partial \psi} = \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial \psi} + \frac{\partial T}{\partial t'}\frac{\partial t'}{\partial \psi}$$

$$\frac{\partial \tau}{\partial t} = \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial t} + \frac{\partial T}{\partial t'}\frac{\partial t'}{\partial t}$$

$$\frac{\partial \tau}{\partial \phi} = \frac{\partial T}{\partial \phi}. \tag{1.8}$$

From (1.5), $\partial\theta/\partial\psi = 1$ and $\partial\theta/\partial t = -\omega$. Also $\partial T/\partial t' = 0$ since the temperature in the r, θ, ϕ system does not vary with time. The general differential equation (1.4) becomes

$$\frac{1}{\sin^2\phi}\frac{\partial^2 T}{\partial \theta^2} + \frac{\rho c_p r_0^2 \omega}{k}\frac{\partial T}{\partial \theta} + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial T}{\partial \phi}\right) - \frac{\sigma e r_0^2}{ks}T^4 = -\frac{aK_s r_0^2}{ks}F(\theta, \phi). \tag{1.9}$$

1.3. Linearization of the differential equation

(1.9) is non-linear. To linearize this equation, the following definition is made,

$$T = T_0(1 + \hat{T}). \tag{1.10}$$

It is assumed that the temperature variation $\hat{T}$, about some reference value T_0, is small compared to unity. Therefore,

$$T^4 \approx T_0^4(1 + 4\hat{T}).$$

Further define

$$\bar{T} = \hat{T} + \tfrac{1}{4}. \tag{1.11}$$

It follows that

$$T = T_0(\tfrac{3}{4} + \bar{T}) \quad T^4 = 4T_0^4\bar{T}$$

$$\frac{\partial T}{\partial \theta} = T_0\frac{\partial \bar{T}}{\partial \theta} \qquad \frac{\partial^2 T}{\partial \theta^2} = T_0\frac{\partial^2 \bar{T}}{\partial \theta^2} \qquad \frac{\partial T}{\partial \phi} = T_0\frac{\partial \bar{T}}{\partial \phi}. \tag{1.12}$$

(1.9) takes the form

$$\frac{1}{\sin^2\phi}\frac{\partial^2 \bar{T}}{\partial\theta^2}+\frac{\rho c_p r_0^2\omega}{k}\frac{\partial\bar{T}}{\partial\theta}+\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial\bar{T}}{\partial\phi}\right)-\frac{4\sigma e r_0^2 T_0^3}{ks}\bar{T}=-\frac{aK_s r_0^2}{T_0 ks}F(\theta,\phi). \quad (1.13)$$

(1.13) is the general linearized equation for the non-dimensional temperature variation $\bar{T}$.

Introducing the constants,

$$b'=\frac{\rho c_p r_0^2\omega}{k} \quad (1.14)$$

$$c'=\frac{4\sigma e r_0^2 T_0^3}{ks} \quad (1.15)$$

$$a'=-\frac{aK_s r_0^2}{T_0 ks} \quad (1.16)$$

yields the following equation

$$\frac{1}{\sin^2\phi}\frac{\partial^2\bar{T}}{\partial\theta^2}+b'\frac{\partial\bar{T}}{\partial\theta}+\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial\bar{T}}{\partial\phi}\right)-c'\bar{T}=a'F(\theta,\phi). \quad (1.17)$$

(1.17) may be reduced to the corresponding equation for a cylindrical shell (which rotates about its geometric axis) by imposing the conditions $\partial\bar{T}/\partial\phi = 0$ and $\phi = \pi/2$.

1.4. *Radiation input function*

One half of the satellite is in darkness while the other half is illuminated by the incoming radiation. At the surface of the satellite, the region of darkness is defined by $\pi \leqq \theta \leqq 2\pi$, $0 \leqq \phi \leqq \pi$, while the illuminated region is defined by $0 \leqq \theta \leqq \pi$, $0 \leqq \phi \leqq \pi$.

The radiation input function $F(\theta, \phi)$ gives at each point, $P(r_0, \theta, \phi)$, the component of incident radiant energy which is normal to the surface of the satellite. For the dark region,

$$F(\theta,\phi)=0.$$

For the illuminated region,

$$F(\theta,\phi)=\bar{q}\,.\,\bar{n}_r$$

where $\bar{q}$ is a unit vector directed negatively along the Y-axis, and $\bar{n}_r$ is a unit vector normal to the surface of the satellite and directed inwardly (Fig. 3). It follows,

$$F(\theta,\phi)=\begin{cases}\sin\theta\sin\phi & 0\leqq\theta\leqq\pi \quad 0\leqq\phi\leqq\pi\\ 0 & \pi\leqq\theta\leqq 2\pi, \quad 0\leqq\phi\leqq\pi.\end{cases} \quad (1.18)$$

2. THE TEMPERATURE DISTRIBUTION

The geometry of the problem indicates a solution in terms of associated Legendre functions. For this reason it is desirable to expand the radiation input in terms of these functions.

2.1. *Expansion of the radiation input function*

Since any piecewise continuous function of θ and ϕ, for $0 \leqq \theta \leqq 2\pi$ and $0 \leqq \phi \leqq \pi$, can be expanded in a series of spherical harmonics [3],

$$F(\theta,\phi)=\sum_{n=0}^{\infty}\left[B_n P_n(\cos\phi)+\sum_{m=1}^{n}(B_n^m\cos m\theta+C_n^m\sin m\theta)P_n^m(\cos\phi)\right]. \quad (2.1)$$

$P_n(\cos\phi)$ is the Legendre polynomial of the first kind, and $P_n^m(\cos\phi)$ is the associated Legendre function of the first kind. The definitions of these functions are

$$P_n(x)=\frac{1}{2^n.n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$P_n^m(x)=(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)$$

where

$$x=\cos\phi.$$

The coefficients of (2.1) are [3]

$$B_n=\frac{2n+1}{4\pi}\int_0^{2\pi}\int_{-1}^{1}F(\theta,x)P_n(x)\,dx\,d\theta$$

$$B_n^m=\frac{2n+1}{2\pi}\frac{(n-m)!}{(n+m)!}\int_0^{2\pi}\int_{-1}^{1}F(\theta,x)P_n^m(x)\cos m\theta\,dx\,d\theta$$

$$C_n^m = \frac{2n+1}{2\pi}\frac{(n-m)!}{(n+m)!}\int_0^{2\pi}\int_{-1}^{1} F(\theta, x) P_n^m(x) \sin m\theta \,dx\, d\theta$$

where $F(\theta, x)$ is found from (1.18) by substitution of $x = \cos\phi$,

$$F(\theta, x) = \begin{cases} (1-x^2)^{1/2}\sin\theta & 0 \leqq \theta \leqq \pi, \; -1 \leqq x \leqq 1 \\ 0 & \pi \leqq \theta \leqq 2\pi, \; -1 \leqq x \leqq 1. \end{cases} \tag{2.2}$$

Utilizing the relations, [4]

$$\int_{-1}^{1} P_n^m(x) P_l^m(x)\, dx = \begin{cases} 0 & , n \neq l \\ \dfrac{2}{2n+1}\dfrac{(n+m)!}{(n-m)!}, & n = l \end{cases}$$

it follows

$$\begin{aligned} C_1^1 &= \tfrac{1}{2} \\ C_n^1 &= 0 \quad n \neq 1 \\ C_n^m &= 0 \quad m \neq 1. \end{aligned} \tag{2.3}$$

Finally, the expansion of $F(\theta, \phi)$ becomes

$$F(\theta, \phi) = \tfrac{1}{2}\sin\theta\sin\phi + \sum_{n=0}^{\infty}\left[B_n P_n(\cos\phi) + \sum_{m=1}^{n} B_n^m \cos m\theta\, P_n^m(\cos\phi)\right] \tag{2.4}$$

where

$$B_n = \frac{2n+1}{2\pi}\int_{-1}^{1} (1-x^2)^{1/2} P_n(x)\, dx \tag{2.5}$$

$$B_n^m = -\frac{2n+1}{2\pi}\frac{(n-m)!}{(n+m)!}\frac{(-1)^m+1}{m^2-1}\int_{-1}^{1}(1-x^2)^{1/2} P_n^m(x)\, dx, \quad m \neq 1 \tag{2.6}$$

$$B_n^1 = 0. \tag{2.7}$$

A few values for B_n and B_n^m were computed:

$$\left\{\begin{array}{ll} B_0 = \frac{1}{4} & B_3 = 0 \\ B_1 = 0 & B_4 = -\frac{9}{256} \\ B_2 = -\frac{5}{32} & B_5 = 0 \end{array}\right\} \tag{2.8}$$

$$\left\{\begin{array}{ll} B_1^1 = 0 & B_4^1 = 0 \\ & B_4^2 = -\frac{1}{256} \\ B_2^1 = 0 & B_4^3 = 0 \\ B_2^2 = -\frac{5}{64} & B_4^4 = -\frac{1}{2048} \\ B_3^1 = 0 & \\ B_3^2 = 0 & \\ B_3^3 = 0 & \end{array}\right\}$$

2.2. *Solution of the differential equation*

For the general solution of (1.17), assume

$$\bar{T} = (\gamma\cos\theta + \beta\sin\theta)\sin\phi + \sum_{n=1}^{\infty} D_n P_n(\cos\phi) + \sum_{n=1}^{\infty}\sum_{m=1}^{n}(D_n^m \cos m\theta + E_n^m \sin m\theta) P_n^m(\cos\phi). \tag{2.9}$$

After a change of variable

$$x = \cos\phi \tag{2.10}$$

(2.9) is introduced into the differential equation (1.17), utilizing the following relations

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_n(x)}{dx}\right] = -n(n+1)P_n(x)$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_n^m(x)}{dx}\right] = -\left[n(n+1) - \frac{m^2}{1-x^2}\right] P_n^m(x).$$

After regrouping it follows that

$$\begin{aligned} &\left\{[b'\beta - (c'+2)\gamma]\cos\theta + \left[-b'\gamma - (c'+2)\beta - \frac{a'}{2}\right]\sin\theta\right\}\sin\phi \\ &+ \sum_{n=0}^{\infty}\{-[c'+n(n+1)]D_n - a'B_n\}P_n(\cos\phi) \\ &+ \sum_{n=1}^{\infty}\sum_{m=1}^{n}\{-[c'+n(n+1)]D_n^m + b'mE_n^m - a'B_n^m\}\cos m\theta\, P_n^m(\cos\phi) \\ &+ \sum_{n=1}^{\infty}\sum_{m=1}^{n}\{-b'mD_n^m - [c'+n(n+1)]E_n^m\}\sin m\theta\, P_n^m(\cos\phi) = 0 \end{aligned} \tag{2.11}$$

(2.11) will be valid for all values of ϕ and θ if

$$D_n = \frac{-a'B_n}{c' + n(n+1)} \qquad (2.12)$$

$$D_n^m = \frac{-a'[c' + n(n+1)]B_n^m}{[c' + n(n+1)]^2 + m^2b'^2} \qquad (2.13)$$

$$E_n^m = \frac{ma'b'B_n^m}{[c' + n(n+1)]^2 + m^2b'^2} \qquad (2.14)$$

$$\gamma = \frac{-a'b'}{2[(c'+2)^2 + b'^2]} \qquad (2.15)$$

$$\beta = \frac{-a'(c'+2)}{2[(c'+2)^2 + b'^2]}. \qquad (2.16)$$

2.3. *Continuity conditions*

It can be seen by inspection that the assumed temperature distribution, $T = T_0(\frac{3}{4} + \hat{T})$ and (2.9), satisfy the continuity conditions,

$$T(r_0, \theta_1, \phi) = T(r_0, \theta_1 + 2\pi, \phi)$$

$$\left(\frac{\partial T}{\partial \theta}\right)_{\theta=\theta_1} = \left(\frac{\partial T}{\partial \theta}\right)_{\theta=\theta_1+2\pi}$$

$$\left(\frac{\partial T}{\partial \phi}\right)_{\theta=\theta_1} = \left(\frac{\partial T}{\partial \phi}\right)_{\theta=\theta_1+2\pi}$$

2.4. *Average temperature*

In the general solution for the temperature distribution, $T = T_0(\frac{3}{4} + \hat{T})$, the reference temperature T_0 appears in the solution explicitly and also implicitly through the constants γ, β, D_n, D_n^m and E_n^m.

This reference temperature may be defined as the average temperature

$$T_0 = \frac{\int_A T\,\mathrm{d}A}{\int_A \mathrm{d}A}.$$

Since $T = T_0(1 + \hat{T})$, it follows that $\int_A \hat{T}\,\mathrm{d}A = 0$.

The heat radiated to space is equal to the energy absorbed and therefore

$$\sigma e \int_A T^4\,\mathrm{d}A \approx \sigma e \int_A T_0^4(1 + 4\hat{T})\,\mathrm{d}A = 4\pi r_0^2 \sigma e T_0^4$$

$$= \int_0^\pi \int_0^\pi aK_s r_0 \sin\theta \sin^2\phi \,\mathrm{d}\phi\,\mathrm{d}\theta = aK_s r_0^2 \pi$$

and

$$T_0 = \left(\frac{aK_s}{4\sigma e}\right)^{1/4}. \qquad (2.17)$$

Charnes and Raynor [1] show that the average temperature for the cylinder is

$$T_0 = \left(\frac{aK_s}{\pi\sigma e}\right)^{1/4}.$$

2.5. *Dimensionless groups*

The constants a', b', c' which appear in the differential equation and solution are related to dimensionless groups

$$\frac{c'}{a'} = -\frac{4\sigma e T_0^4}{aK_s}. \qquad (2.18)$$

Substituting (2.17) into (2.18) gives

$$\frac{c'}{a'} = -1. \qquad (2.19)$$

A new dimensionless group called the thermal radius can be defined

$$R = \frac{r_0^2}{ks}(4\sigma e a^3 K_s^3)^{1/4}. \qquad (2.20)$$

Then

$$c' = R \qquad (2.21)$$

$$a' = -R. \qquad (2.22)$$

A thermal angular velocity can be defined

$$\omega^* = \frac{\alpha}{r_0^2} \qquad (2.23)$$

where

$$\alpha = \frac{k}{\rho c_p}.$$

Further, a non-dimensional angular velocity can be defined

$$\omega_0 = \frac{\omega}{\omega^*}. \qquad (2.24)$$

From (1.14) it is seen that

$$b' = \omega_0. \qquad (2.25)$$

2.6. *Solution summary*

In non-dimensional form the temperature distribution is given by

$$\frac{T(r_0, \theta, \phi)}{T_0} = \frac{3}{4} + (\gamma\cos\theta + \beta\sin\theta)\sin\phi + \sum_{n=0}^{\infty} D_n P_n(\cos\phi) + \sum_{n=1}^{\infty}\sum_{m=1}^{n}(D_n^m \cos m\theta + E_n^m \sin m\theta)P_n^m(\cos\phi) \qquad (2.26)$$

where

$$T_0 = \left(\frac{aK_s}{4\sigma e}\right)^{1/4} \quad (2.27)$$

$$\gamma = \frac{R\omega_0}{2[(R+2)^2 + \omega_0^2]} \quad (2.28)$$

$$\beta = \frac{R(R+2)}{2[(R+2)^2 + \omega_0^2]} \quad (2.29)$$

$$D_n = \frac{RB_n}{R + n(n+1)} \quad (2.30)$$

$$D_n^m = \frac{R[R + n(n+1)]B_n^m}{[R + n(n+1)]^2 + m^2\omega_0^2} \quad (2.31)$$

$$E_n^m = -\frac{mR\omega_0 B_n^m}{[R + n(n+1)]^2 + m^2\omega_0^2}. \quad (2.32)$$

The dimensionless parameters R and ω_0 are defined by (2.20), (2.23) and (2.24). The coefficients B_n and B_n^m are given by (2.5), (2.6) and (2.7). Some values of B_n and B_n^m are given by (2.8).

3. DISCUSSION OF THE RESULTS

The non-dimensional local temperature T/T_0 of the satellite is a function of two non-dimensional parameters R (thermal radius) and ω_0 (non-dimensional angular velocity).

3.1. *Limiting cases*

The temperature distribution for the following cases will be considered.

Case 1. Non-rotating sphere

$$\frac{T}{T_0} = \tfrac{3}{4} + \beta \sin\theta \sin\phi + \sum_{n=0}^{\infty} D_n P_n(\cos\phi) + \sum_{n=1}^{\infty}\sum_{m=1}^{n} D_n^m \cos m\theta\, P_n^m(\cos\phi) \quad (3.1)$$

$$\beta = \frac{R}{2(R+2)} \qquad D_n = \frac{RB_n}{R+n(n+1)}$$

$$D_n^m = \frac{RB_n^m}{R+n(n+1)}. \quad (3.2)$$

The maximum temperature occurs at $\phi = \pi/2$ and $\theta = \pi/2$. The minimum temperature occurs at $\phi = \pi/2$ and $\theta = (3/2)\pi$.

The difference between the maximum and minimum temperature is

$$\frac{T_{\max} - T_{\min}}{T_0} = \frac{R}{R+2}. \quad (3.3)$$

Case 2. Infinitely high velocity

$$\frac{T}{T_0} = \tfrac{3}{4} + \sum_{n=0}^{\infty} D_n P_n(\cos\phi). \quad (3.4)$$

The temperature is independent of the longitude θ. The temperature distribution is shown in Fig. 4 for the numerical example that follows.

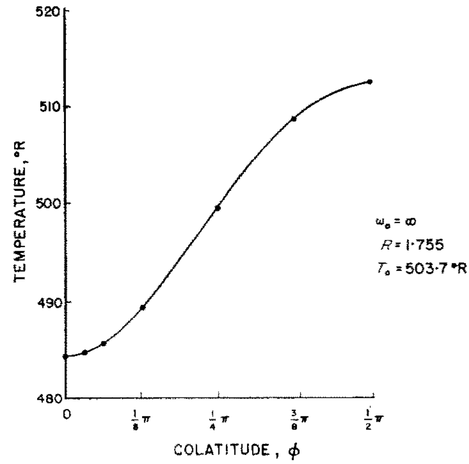

FIG. 4. Quasi steady temperature distribution for $\omega_0 = \infty$.

In general, the positions of maximum and minimum temperature with respect to longitude θ are found by requiring that $\partial T/\partial\theta = 0$. Applying this to (2.26) yields

$$(\gamma \sin\theta - \beta\cos\theta)\sin\phi + \sum_{n=1}^{\infty}\sum_{m=1}^{n} m(D_n^m \sin m\theta - E_n^m \cos m\theta)P_n^m(\cos\phi) = 0. \quad (3.5)$$

For given values of co-latitude ϕ, (3.5) becomes a transcendental equation in θ. Due to the complexity, only the leading term of the series was considered. Thus

$$\gamma\sin\theta - \beta\cos\theta = (6E_2^2 \sin\phi)\cos 2\theta - (6D_2^2\sin\phi)\sin 2\theta. \quad (3.6)$$

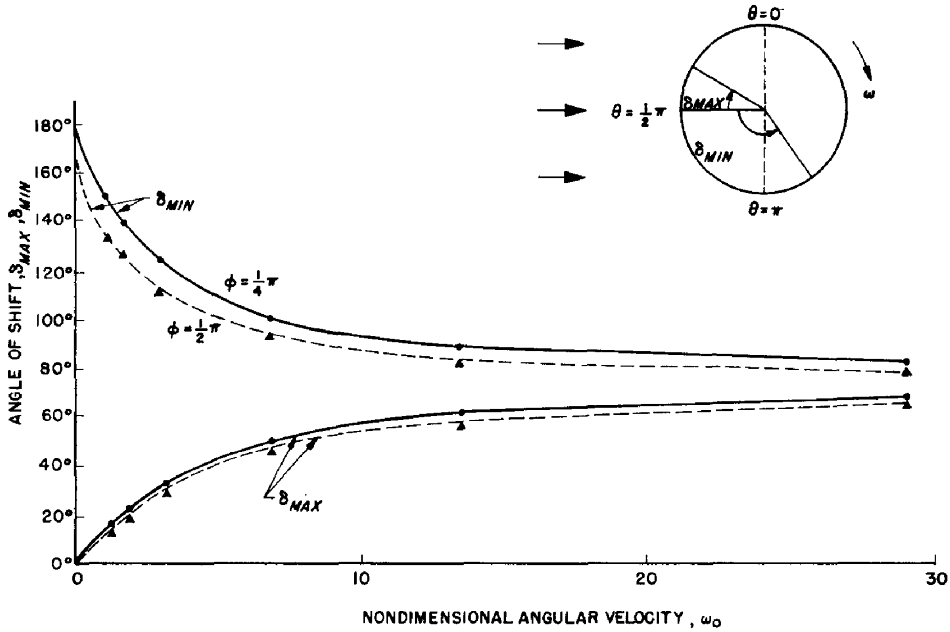

FIG. 5. Positions of maximum and minimum temperature.

The results for the numerical example considered are plotted in Fig. 5. As $\omega \to \infty$ the values for "virtual" maximum and minimum temperature can be computed from the relation

$$\sin\theta \sin\phi + \sum_{n=1}^{\infty} \sum_{m=1}^{n} 2B_n^m \cos m\theta \, P_n^m(\cos\phi) = 0.$$

Table 1. *Asymptotic positions of maximum and minimum temperature* ($\omega_0 = \infty$)

ϕ (radians)	θ_{max} (degrees)	θ_{min} (degrees)
0	0	180
$\pi/36$	4·5	175·5
$\pi/18$	8	172
$\pi/8$	15	165
$\pi/6$	17·5	162·5
$\pi/2$	18·5	161·5

The results are presented in Table 1. It should be noticed that both maximum and minimum temperature occur on the "sunny" side. These values are independent of all physical parameters.

3.2. *Relation of pole temperature to average temperature*

An interesting aspect of the temperature distribution is the behavior of the temperature at the satellite poles, $\phi = 0, \pi$. At the poles, (2.26) reduces to

$$\frac{T_{\text{pole}}}{T_0} = \tfrac{3}{4} + \sum_{n=0}^{\infty} \frac{RB_n}{R + n(n+1)}.$$

Since $B_0 = \frac{1}{4}$, this equation becomes

$$\frac{T_{\text{pole}}}{T_0} = 1 + \sum_{n=1}^{\infty} \frac{RB_n}{R + n(n+1)}. \quad (3.7)$$

It should be noticed that the pole temperature is independent of angular velocity.

Now consider the two special cases. For $R = 0$

$$\frac{T_{\text{pole}}}{T_0} = 1$$

and as R becomes very large,

$$\lim_{R\to\infty} \frac{T_{\text{pole}}}{T_0} = 1 + \sum_{n=1}^{\infty} B_n.$$

Using the values of the coefficients B_n, it follows

$$\lim_{R\to\infty} \frac{T_{\text{pole}}}{T_0} = 0{\cdot}809.$$

Thus

$$0{\cdot}809 \leqq \frac{T_{\text{pole}}}{T_0} \leqq 1. \tag{3.8}$$

Thus, for all satellites of the type considered, the pole temperature will be within 80 per cent of the average temperature of the satellite.

3.3. *Error analysis*

Under steady state conditions there will be a balance between the total energy absorbed and radiated from the satellite; thus,

$$\int_A aK_s F(\theta, \phi)\, \mathrm{d}A = \int_A \sigma e \tilde{T}^4\, \mathrm{d}A \tag{3.9}$$

where $\tilde{T}$ is the true temperature, as would be found from the solution of the general non-linear differential equation. If the temperature, T, given by the linearized equation were used in the overall heat balance, there would be an error ϵ_Q such that

$$\int_A aK_s F(\theta, \phi)\, \mathrm{d}A = (1 - \epsilon_Q) \int_A \sigma e T^4\, \mathrm{d}A. \tag{3.10}$$

From (3.10) and definition of T_0 follows

$$\epsilon_Q = 1 - 4\pi \left[\int_0^{\pi}\int_0^{2\pi} \left(\frac{T}{T_0}\right)^4 \sin\phi\, \mathrm{d}\phi\, \mathrm{d}\theta\right]^{-1}. \tag{3.11}$$

Due to the linearization, the radiation effect has been slightly moderated; therefore, the true temperature would always be less than the corresponding temperature found from the linearized equation. Thus a temperature error function $\epsilon_T(\theta, \phi)$ can be defined

$$T = \tilde{T}[1 + \epsilon_T(\theta, \phi)]. \tag{3.12}$$

From (3.9) and (3.10) follows

$$\int_A \tilde{T}^4\, \mathrm{d}A = (1 - \epsilon_Q) \int_A T^4\, \mathrm{d}A. \tag{3.13}$$

Substitution of (3.12) into (3.13) and calling $\bar{\epsilon}_T$ the average value of ϵ_T results in

$$\bar{\epsilon}_T = \frac{\int_A \epsilon_T \tilde{T}^4\, \mathrm{d}A}{\int_A \tilde{T}^4\, \mathrm{d}a} = \frac{\epsilon_Q}{4(1 - \epsilon_Q)}. \tag{3.14}$$

The largest errors are incurred when temperatures deviate greatly from the average temperature T_0. This occurs for $\omega_0 = 0$. Using the data for the numerical example that follows, ϵ_Q was determined through numerical integration of (3.11). It was found that $\epsilon_Q = 0{\cdot}10$ resulting in $\bar{\epsilon}_T = 2{\cdot}8$ per cent for $\omega_0 = 0$.

3.5. *Numerical example*

In order to compute the non-dimensional temperature distribution T/T_0, the following values for the physical parameters were selected as representative of an aluminum alloy satellite

radius	r_0	$= 1$ ft
wall thickness	s	$= 1/200$ ft
thermal conductivity	k	$= 100$ Btu/ft h degR
thermal diffusivity	a	$= 3$ ft²/h
solar constant	K_s	$= 442$ Btu/ft²h
radiation constant	σ	$= 0{\cdot}1717 \times 10^{-8}$ Btu/ft²h degR⁴.

Furthermore, the surface of the satellite will be considered as "thermally black" in order to maximize the radiation effects. From the above values result

$$R = \frac{r_0^2}{ks}(4\sigma e a^3 K_s^3)^{1/4} = 1{\cdot}755$$

$$\omega^* = \frac{a}{r_0^2} = 3 \text{ rad/h}$$

$$T_0 = \left(\frac{aK_s}{4\sigma e}\right)^{1/4} = 503{\cdot}7 \text{ degR}.$$

In order to demonstrate the role of angular velocity, a range of values of ω were chosen; these correspond to values of the non-dimensional parameter $\omega_0 = \omega/\omega^*$.

To compute the temperature distribution, (2.26) was approximated with terms through $n = 4$ and $m = 4$. From this approximation, the temperature distribution as a function of longitude θ was determined for four positions of co-latitude, $\phi = 0, \pi/8, \pi/4, \pi/2$. Values of $\omega = 0$, $8{\cdot}710$ and $87{\cdot}10$ rad/h were chosen; these correspond to values of $\omega_0 = 0$, $2{\cdot}903$ and $29{\cdot}03$ respectively. These data are plotted in Figs. 6, 7 and 8. The temperature at the pole $\phi = 0$ is independent of longitude θ and angular

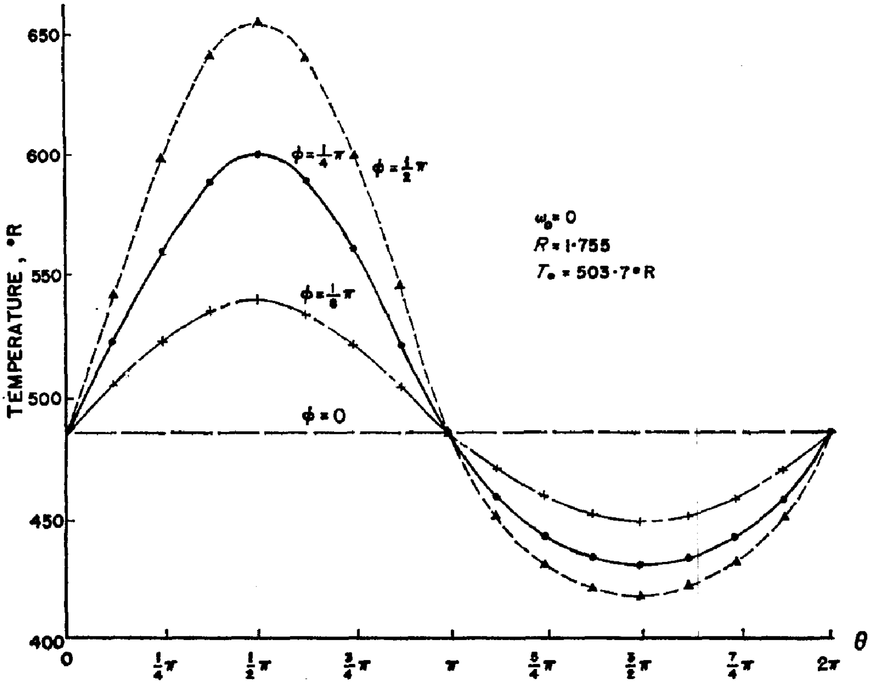

FIG. 6. Quasi steady temperature distribution for $\omega_0 \neq 0$.

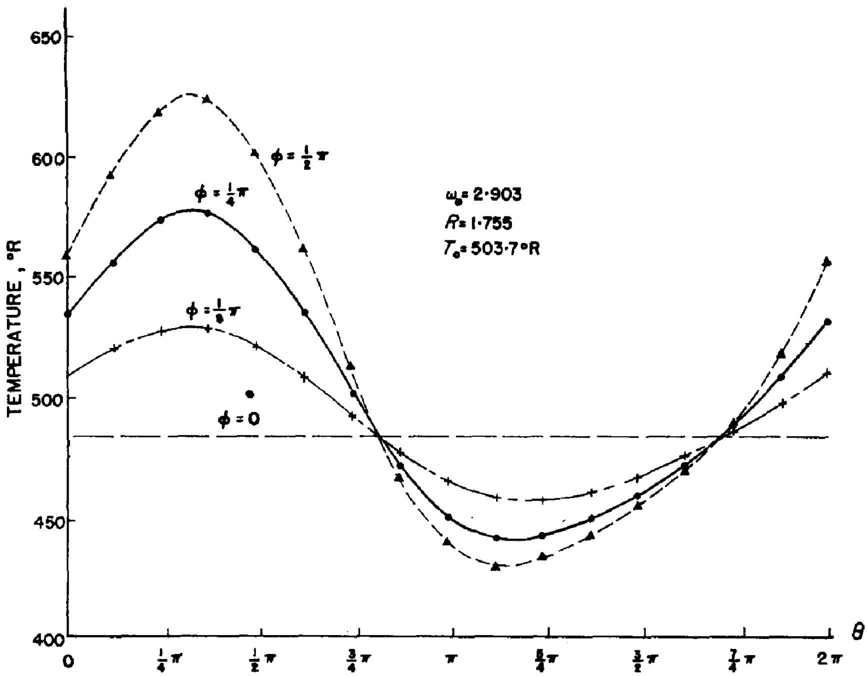

FIG. 7. Quasi steady temperature distribution for $\omega_0 = 2{\cdot}903$.

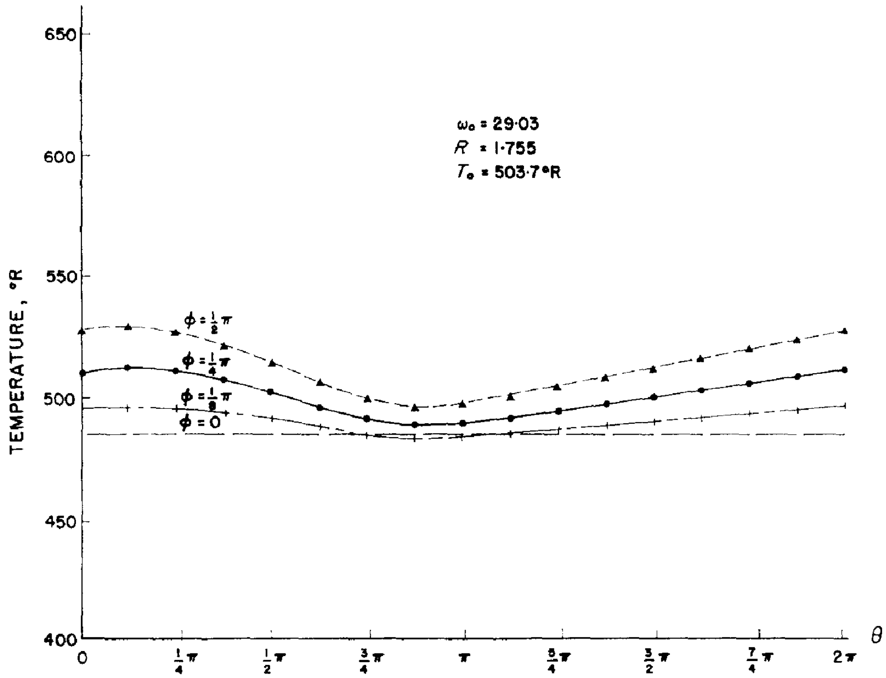

FIG. 8. Quasi steady temperature distribution for $\omega_0 = 29{\cdot}03$.

velocity ω, thus it appears as the same horizontal line in these figures. Comparison of Figs. 6, 7 and 8 shows, that as ω_0 increases, the temperature distributions at $\phi = \pi/8$, $\pi/4$, $\pi/2$ flatten out, while the positions of maximum and minimum temperature shift in the direction of rotation, Fig. 5.

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Résumé—Cet article étudie ladistribution de températures d'un satellite sphérique à paroi mince soumis au rayonnement solaire. Le satellite tourne autour d'un axe perpendiculaire au rayonnement incident. L'équation différentielle linéaire de la distribution de températures est résolue en fonction des polynômes de Legendre et de leurs fonctions associées. Les résultats montrent l'influence de la vitesse angulaire sur la distribution des températures. Un exemple numérique est donné et les erreurs dues à la linéarisation des équation sont évaluées.

Zusammenfassung—An einem dünnwandigen kugelförmigen Satelliten wird die Temperaturverteilung bei Sonneneinstrahlung behandelt. Der Satellit rotiert um eine zur einfallenden Strahlung senkrechte Achse. Die linearisierte Differentialgleichung der Temperaturverteilung ist in Form von Legendre-Polynomen und verwandten Legendre-Funktionen gelöst. Die Ergebnisse zeigen den Einfluss der Winkelgeschwindigkeit auf die Temperaturverteilung. Ein numerisches Beispiel und eine Fehleranalyse sollen die Abweichungen infolge der Linearisierung zeigen.

Аннотация—В данной статье рассматривается распределение температуры в тонких стенках спутника, имеющего форму шара и испытывающего солнечную радиацию. Спутник вращается вокруг оси, перпендикулярной к падающему излучению. Линеаризованное дифференциальное уравнение для распределения температуры выражается через полиномы Лежандра и присоединенные функции Лежандра. Результаты подтверждают влияние угловой скорости на распределение температуры.

Приводится численный пример, а также анализируются ошибки вычисления с тем чтобы оценить погрешности за счет процесса линеаризации.